Efficient mathematics

1. Purpose

Over the centuries, mathematics has been promoted and valued for many reasons, including tradition, religion, as part of philosophy, for aesthetic reasons, as a companion to science, and as a pure intellectual discipline.

But, surely, the original and most important reason is its useful contribution to society. We, as a society, invest in mathematics primarily because it is *useful*. For example, we spend a lot of time and effort teaching some mathematics to children. We support teaching of mathematics in universities. We fund mathematical research.

Consequently we can, quite reasonably, look at the value and cost of mathematics, and consider the efficiency of mathematics as a contributor to society.

2. Some major contributions

What value has mathematics provided over the centuries? Could we cope without mathematics? Let's consider some of the major branches of mathematics.

2.1 Counting and basic arithmetic

Numbers allowed the invention of calendars and dates. Early civilizations were able to count days and begin to predict important times such as floods, rains, times for harvesting, and phases of the moon.

Imagine life without the ability to make arrangements with people and put them in a diary. Without this precision we would be reduced to agreements like 'I'll see you when the rains come' and 'Let's do lunch the day after the next full moon, whenever that is.'

It would be hard to keep in touch with anyone you weren't living very close to and seeing daily.

Keeping time is another invention that requires numbers and counting. Without it we must agree to have lunch 'later' or to meet 'at dusk.' Measuring time also allowed us to understand a lot about movement, velocity, and speed.

Numbers let us agree prices, use money, and make contracts. Without all this, every day would be a relentless grind of bartering. Imagine receiving your wages daily in kind and having to challenge whenever they seem a bit mean. 'Boss, that's not as much as I was expecting and those turnips look a bit wrinkly.'

Numbers also give us the ability to delegate work to people and leave them unobserved to get on with it. Imagine you had a factory with stocks of valuable items. It would be much harder to stop employees just taking your stuff whenever they wanted to if you could not count your stock from time to time, and could not reconcile the stock you have with the money you should have received for what has been sold.

A lot of the planning we do requires counting and basic arithmetic. We look at quantities of goods, raw materials, labour, and the time needed to do tasks. Without this our plans would feature 'some work' and big projects would take 'a long time.'

Counting and basic arithmetic bring much convenience and efficiency to our lives.

2.2 Geometry

Geometry is another branch of mathematics with obvious and ancient applications.

Geometry enabled fields to be laid out again after they were covered with silt from the Nile. It allowed mapping, architecture, town layouts, division of land, and navigation.

In modern times it is required endlessly in the development of software for computer graphics and medical scanning.

The ability to understand and produce straight lines, circles, ellipses, parabolas and other useful and attractive shapes means our world is more efficient, comfortable, and attractive than it otherwise would be.

2.3 Probability theory

We could easily do without a betting 'industry' but life insurance is useful. Before mathematics was applied to the problem, life insurance companies frequently collapsed because they could not estimate their future liabilities. Today they collapse much less often.

2.4 Symbolic logic

Symbolic logic has facilitated the development of microchips, calculators, and computers. A lot of this might have been achieved without mathematical techniques but compact notations and useful theorems make the complex logic easier to understand. Life without computers is possible, but not something we would welcome.

2.5 Calculus

Calculus is one of those topics that marks the beginning of the mathematics most people don't need to know. Most students at school who don't really get on with mathematics drop the subject before they reach calculus.

But calculus and the dynamical models it works so well with have been very useful in the development of science and technology.

Calculus allows us to find the areas and volumes of curved shapes, to find an optimum design (in situations we can measure and understand well), to plan space journeys, and to forecast the weather with huge simulations.

It helps us find streamlined shapes for things, create bridges and very tall buildings that are stable in high winds, check that buildings are structurally sound but still economical, and design electrical equipment and communication systems.

If you can't do calculus it is unlikely anyone will every ask you to, but if you can do it then there are useful, well paid jobs where it is needed.

2.6 Summary of value

In summary, our society could cope without mathematics but life would be much harder, less organized, less efficient, less attractive, and less healthy. There is also doubt as to whether the earth could sustain so many of us without the efficiency that mathematics allows.

The situation for individuals is less clear cut. If you can't count or do basic arithmetic then life is difficult and embarrassing. If you can't do more advanced mathematics then you can live quite happily, but you can't do jobs that use mathematics, such as many jobs in engineering and science, even if you could get an interview and offer.

3. Today's efficiency

Is mathematics contributing to society efficiently today, or are there obvious problem areas? Most people already know from personal experience that there are some big issues.

3.1 Learning in primary school

Slow learning of number facts: There are a little over 300 basic number facts that we try to teach children before they reach the age of 11. Some children pick them up quickly and easily and get the whole lot in a few weeks. Yet many other children still have not mastered those facts after 7 years of teaching, usually with an hour of arithmetic a day plus a small amount of homework.

Because we still have not found an efficient, reliable way to teach basic number facts and their use, many people's first experience of mathematics is unhappy. For them it is a struggle, with little or no progress from one year to the next. It's boring, embarrassing, discouraging – and much easier to make excuses.

Unrealistic exercises: One excuse is that maths is not important. This is an excuse that makes a lot more sense when you look at typical problems in school mathematics.

Some are realistic, meaning that you really might need to know how to solve them. Others are ridiculous, contrived situations you would never have to solve except in a puzzle book.

Teacher: 'If Anne is two years older than Bob, but twice as old as her dog Spot, and Spot is three years old, how old are Anne and Bob?'

Student: 'Since you obviously know how old they are why don't you just tell me?'

Time students spend on problems they will never need in the real world is time they could have spent learning the same techniques on problems that they might face.

3.2 Encouraging good mathematicians

Focus on pointless puzzles: Children who do well at maths often get encouragement, which is good. Unfortunately, some of that encouragement involves toiling with pointless problems.

In the UK, for example, the UK Mathematics Trust runs an annual competition for outstanding young mathematicians, starting in primary school. Overall, this is a good thing and draws out the top performers.

However, nearly all the problems they set are problems with no hope of practical application. This steers the students towards less useful techniques and sends the unhelpful message to everyone that maths is really about doing hard but useless puzzles – a sort of high-brow Sudoku.

Popular science books and websites aimed at people of all ages who like mathematics tend to focus on a narrow range of topics, mostly of little practical use. They love number theory problems, imaginary numbers, stuff about infinity, and recreational mathematics that only occasionally gets useful.

3.3 Learning at secondary school

More unrealistic exercises: The mistake of setting pointless problems only gets more common at secondary school as algebra is introduced to children.

Textbook: 'A rectangular swimming pool is $3x - 5$ metres long and x metres wide. Its surface area is 250m^2. How long is the pool?[1]'

Student thinking: 'If someone asked me this in real life I'd tell them to stop wasting my time and just say how big the pool is. This is pointless.'

The intention of the problem is to practise setting up and solving quadratic equations, but surely there are more realistic problems that can give this opportunity?

Problems like this are very common and should be avoided completely because of the unhelpful message they send: that maths is about pointless but difficult puzzles.

A less obvious example is the SUVAT equations for motion under constant acceleration. This feels like an applied maths topic but air resistance always has to be ignored. This means that in most cases the equations would not give an answer accurate in real life, and that's disappointing and unsettling to any student who thinks about it.

No teaching of practical application of mathematics: The whole task of checking equations against reality and deciding what is

[1] 25 metres.

good enough is ignored. It would not be hard to show equations that are more accurate when air resistance is a factor, though young students cannot do much with them other than plug in numbers and calculate predictions.

Anachronistic syllabi: Although the typical syllabus for secondary school mathematics contains many fundamental techniques, some of the techniques, and the emphasis given, are the result of history. It should instead be the result of studying what mathematics people can use in the modern world, with its computers and big data.

For example, secondary school mathematics has a lot of material on polynomials but is that still the most useful area to be expert? What about data mining methods, Monte Carlo simulation, Euler's method for exploring differential equations, numerical integration, or iterative optimisation methods? At secondary school level it would not be possible to go into the theory of these methods in detail but it would be possible to experience using them (on computers of course) and know when they are helpful. Since most people would be users of the software involved rather than developers of it, this would be reasonable.

3.4 Learning at university

Slow to reach applications: Students on undergraduate degree courses for mathematics in the UK typically do get the opportunity to experience modern computerised methods that are useful in real world applications. However, not before a lot of other stuff has been worked through first.

The usual degree course structure starts with topics perceived as 'core' or 'foundational', and adds options that are more obviously applied towards the end of year 2 and through year 3. There may also be projects each year where some element of applied mathematics and computer use is included.

However, those 'core' topics tend to be pieces of pure mathematics and even the applied topics are tackled in a way that emphasises definitions, abstractions, and proofs before eventually looking at a (simplified) real problem.

It is also noticeable that the topics are presented in an order rather similar to the historical sequence in which they were invented. You have to get through a lot of Euler, Lagrange, Fermat, and Laplace before you get to work named after someone born as recently as the 20th century.

Topics that could be useful are presented as if applications are secondary. For example, most of number theory (i.e. properties of numbers) focuses on properties that are not very useful in the real world. (They may be useful for showing other useless properties, so mathematicians often see them as useful, even though they are not useful in real world situations.)

Number theory does have one, relatively recent, application that has saved it from being complete useless, which is its application to modern encryption methods for computer data. If you want to learn about it within a mathematics degree you will probably need to wait until your third year and then may need to sign up to a module on 'number theory' rather than 'encryption'.

Measure theory is another example of a topic that involves a lot of hard work for meagre practical reward. The theorising goes on and on but the payoff at the end is the ability to integrate some obscure and useless functions that could not be integrated by other methods.

Failure to organize mathematics efficiently: Abstract algebra again illustrates this problem but also another. Abstract algebra looks at the properties of sets of mathematical objects when subject to mathematical operations (e.g. Real numbers under addition and subtraction).

There are two obvious ways this could be a very useful topic.

First, it could be used to increase the student's ability to develop and explore mathematical approaches to practical problems. For example, the student could learn to define objects and operations that are chosen to model things in the real world, then use abstract algebra knowledge to more quickly sketch out the properties of the mathematical system created.

Second, the various definitions could be structured into one or more hierarchies so that it is easy to see the inheritance of properties from more general definitions to more specific.

Neither of these two ideas is delivered in a typical course. Instead, abstract algebra is a nightmare of confusable definitions of bizarre phrases, usually involving homework that proves pointless properties of mathematical systems with no obvious use.

Preference for publishing pure mathematics: Those students who stay in academia and become teachers and researchers in mathematics write books and papers for journals. Publication is essential for academic survival and success.

The publications that gain most respect are, largely, pure mathematics with no apparent applications. To illustrate this, here are the titles of the articles appearing in the January 2017 edition of the world's highest impact mathematical journal (according to Wikipedia), the Annals of Mathematics:

- 'Cyclic surfaces and Hitchin components in rank 2
- New G_2-holonomy cones and exotic nearly Kähler structures on S^6 and $S^3 \times S^3$
- Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps
- Quasidiagonality of nuclear C^*-algebras
- Lyapunov exponents for random perturbations of some area-preserving maps including the standard map
- A sharp counterexample to local existence of low regularity solutions to Einstein equations in wave coordinates
- Progression-free sets in $\mathbb{Z}_4^n$ are exponentially small
- On large subsets of $\mathbb{F}_q^n$ with no three-term arithmetic progression
- Erratum to "Isotopies of homeomorphisms of Riemann surfaces"'

Do you see any references to anything clearly applicable to a real-world problem? How about references to a medical application, a business problem, an issue in voting systems perhaps, or cryptography?

In case you are thinking that the article mentioning Einstein equations is some useful piece of physics, here is the abstract:

'We give a sharp counterexample to local existence of low regularity solutions to Einstein equations in wave coordinates. We show that there are initial data in H^2 satisfying the wave coordinate condition such that there is no solution in H^2 to Einstein equations in wave coordinates for any positive time. This result is sharp since Klainerman-Rodnianski and Smith-Tataru proved existence for the same equations with slightly more regular initial data.'

Note that this is concerned with theoretical possibilities and not with whether they are frequently occurring in practice, or could be created on purpose in order to achieve some useful outcome.

This was not a special edition.

Compare those titles to the contents of the journal of Operations Research for the same month. (Also not a special edition.)

- 'Supply Function Equilibrium with Taxed Benefits
- Robust Product Line Design
- Optimizing Performance-Based Internet Advertisement Campaigns
- Cross-Selling Investment Products with a Win-Win Perspective in Portfolio Optimization
- Information Elicitation and Influenza Vaccine Production
- Technical Note—Path-Dependent and Randomized Strategies in Barberis' Casino Gambling Model
- Intertemporal Pricing Under Minimax Regret
- Auctions with Dynamic Costly Information Acquisition
- Capacity Investment with Demand Learning
- Fare Evasion in Transit Networks
- The Travelers Route Choice Problem Under Uncertainty: Dominance Relations Between Strategies
- Percentage and Relative Error Measures in Forecast Evaluation
- 0/1 Polytopes with Quadratic Chvátal Rank
- Optimal Resource Capacity Management for Stochastic Networks

- Impact of Delay Announcements in Call Centers: An Empirical Approach
- Rank Centrality: Ranking from Pairwise Comparisons'

The Impact Factor of journals is a measure of how much academic interest they spark, not their impact in the real world. The Impact Factor of the Annals of Mathematics was 3.236 in 2014. For Operations Research it was just 1.743.

Poor mathematical writing: In addition to the baffling, useless output of many mathematical specialists there are innumerable papers generated by scientists that feature some mathematics. Much of this is also very hard to read and often because of poor writing, including confusing or illogical notation.

(I have written a guide to writing mathematics clearly that identifies a large number of common mistakes to avoid. See Leitch, 2009.)

3.5 Using maths at work

Under-use of mathematics: Overall, mathematics is under-used in the workplace. Surveys on the use of management methods tend to put classic management science methods involving quantitative techniques at the very bottom of the popularity league, with hardly any organizations using them.

Mathophobia: I have experienced first-hand the difficulty of trying to get projects going when an element of mathematics is involved. In my case it was clear that several key people felt that they were at risk if their organization started to value analytical skills more highly. They did their best to undermine progress.

This is not surprising when you consider how many people have a hard time with mathematics at school and that many of these people have forgotten what little they did learn all those years ago.

People are also willing to put up with slightly mathematical but obviously wrong approaches rather than make the tiny effort needed to learn a bit more and fix the maths.

Confusing documentation of good software: Another quite different reason for under-use of mathematics in the workplace is that many of the software tools available, while excellent in many ways, are poorly documented.

Take R for example. This is free software for doing statistics and statistical programming. It is supported by a huge academic community and hundreds of packages can be downloaded and installed within it, free of charge and with little hassle. It's fantastic.

The software and every package are documented extensively and all packages are documented in a standardised way.

Unfortunately, that standardised way leads to baffling documentation. There is no requirement for an overview of how the various commands are designed to be used. The commands are simply documented in alphabetical order.

Another problem is that the authors usually assume the reader is another technical expert.

Similarly, WinBUGS is a great piece of software for fitting models to data, but here again the explanation that comes with it leaves a lot to be desired.

These are just two excellent packages that I know about, not packages I have picked out because of particularly confusing documentation.

The underlying issue is that these tools and their documentation are aimed at people who will dedicate their lives to mastering the techniques and packages involved. That means university students and professional academics, plus some professional scientists.

4. The digital shift

One factor underlying many of the issues described above is, I think, an understandable failure to adapt quickly to the changes caused by digital computers.

Until the mid-twentieth century, if you needed to do some calculations you probably had to do them yourself, with a pen and paper, and perhaps some log tables, or maybe a slide rule for approximate answers.

Calculations were very time consuming, tiring, and difficult to do reliably. Naturally, people tried to avoid them and the way to do that was to:

- focus on simple models that you can do symbolic maths with e.g. simple differential equations;
- focus on deriving simple laws and conclusions that can be applied without huge calculations;
- prefer formulae that give you the answer in one go rather than using iterative methods to search for the answer; and
- expand the range of models you can use, making them slightly more complex and realistic, by developing increasingly specialised and un-obvious techniques for manipulating the models symbolically.

With this in mind, education in mathematics has been based on the idea that you need to fully understand and be able to do all the steps of any mathematical technique you might want to use.

Modern software has fundamentally changed the situation. We now have a toolbox of techniques (supported by software) that can do miracles with models of extraordinary complexity and realism, by using iterative numerical methods. Instead of clever symbolic manipulation by people we can use massive number crunching by machines controlled by software to do things that would be impossible symbolically.

These techniques include the following:

- Euler's method for simulating systems specified by differential equations (including equations nobody can solve), and improvements on this method.
- Monte Carlo simulation for calculating the combined effect of multiple uncertainties, and for many other tasks.
- Markov Chain Monte Carlo methods for applying modern Bayesian methods to fit models of nearly any realistic form to nearly any real data.
- An array of methods for mining data, such as Support Vector Machines, neural networks, and symbolic regression.
- Another array of methods for finding optimal or near optimal solutions to problems, ranging from the Newton-Raphson method to simulated evolution.
- Response surface methods for capturing the relationships between variables.
- Finite element methods for modelling forces acting on objects.
- Conjoint analysis – a set of techniques for eliciting values from people.

This means that, today, we can focus on building models that are realistic, rather than models that are simple enough to work with by hand.

It also means that using powerful mathematics does not require us to do every calculation ourselves. Most people only need to know how to use the tools. Only a few people need to know how to make them. Most people can avoid the work needed to understand the methods in detail and prove facts about their performance and limitations.

There is no need to hold back techniques like the ones listed above for advanced university study because most people only need to know how to use them, not prove them.

Once we adapt to this new reality it should be possible for young people to experience, while still at secondary school, the power of realistic, applied mathematics (done for them by software they control). This is instead of toiling painfully to carry out methods that are often too simple to be accurate in real applications.

5. Improving efficiency

The opportunities for improvement will be considered now under the following headings:

- Invention
- Sharing (writing and teaching)
- Application

5.1 Invention

Useful areas: Mathematical research and development in universities as well as outside should focus on useful applications of mathematics. The ratio of applied to pure mathematics should be about 90:10. That doesn't just mean areas where applications are possible in theory; it means focusing on specific problems and solutions that will be useful in the real world.

Academics often like to say that this idea is misguided because some famous and important applications of mathematics have arisen from pure mathematics. 'Look at cryptography!' they may say.

It is surely obvious that useful mathematics is more likely to be developed if people *try* to develop useful mathematics than if they leave it to chance. The preponderance of absurdly specialised theoretical papers by academics and disappointingly low use of mathematics in business and government suggest that the emphasis needs to be changed. That change should be a large swing towards applications.

Realism and general tools: The emphasis should be on developing methods and software tools that allow realistic models to be made and used, preferably applicable to a wide range of models in an application area.

This is exemplified by Markov Chain Monte Carlo methods. These allow a wide range of statistical models to be fitted to data using modern Bayesian methods.

Modernization: More resources should be dedicated to modernizing notation, language, and methods. Mathematics as taught today is changing very, very slowly. There is notation that is often used (and taught in schools) that is ambiguous and confusing but dates back to the original inventors of the techniques, centuries ago.

5.2 Writing

Avoid the generalization syndrome: This syndrome is one of the reasons mathematics is so hard to learn and use. Many famous techniques today were originally invented to tackle a specific practical problem. However, mathematicians then seem to have felt that their work would be more impressive and more generally applicable if the method was generalized and presented as a grand theory, without mentioning the original reason for inventing the methods.

Add to this a tendency to drag in related theory to give a sense of carefully worked out foundations, and you soon have an impressive book. Unfortunately, it requires the reader to have an extensive background in pure mathematics and then plough through chapter after chapter of theoretical preliminaries and remarks before, finally, being able to find and understand the useful method that started it all.

The abstraction and stacking up of theory should be avoided. Methods, applications, and generalizations can be kept.

Write to be applied: Above all, more mathematics should be written so that techniques can be read about, understood, and applied without having to go through months of study of preliminaries.

Write clearly: Mathematics could be made radically easier to understand, learn, and use if it was written more clearly. That could be pursued by teaching clear mathematical writing more intensively, using clarity as a more important factor in deciding what gets published, and developing software tools to assess some of the basics of mathematical clarity.

Use consistent notation: Use of modern, consistent notation would help with clarity and automation.

Clarity makeovers: There exists a vast back catalogue of potentially useful but badly explained mathematics. Modern mathematicians should be encouraged and rewarded for rewriting those in a more accessible, more applicable style. By this I don't mean entertaining popular science. I mean papers and books that can be understood, often with supporting software and examples.

5.3 Teaching

Number fact teaching methods: Efficient methods for teaching basic number facts to children must be found. A far more energetic and rigorous research effort is needed, focused on this particular challenge. This task, more than any other, cuts many people out of mathematics from an early age.

Real problems for exercises: All text books and websites used by schools and universities should be purged of problems nobody ever needs to solve. Let all exercises be developing a simple mechanical skill or solving types of problem that might be encountered in real world applications of mathematics.

No more problems where Anne is twice as old as Bob's dog will be in three years. No more rectangular gardens with sides of $2x + 3$ by $3x$ metres!

Modern school syllabi: School mathematics at secondary school level should be modernized to focus more on modelling skills and computer tools, and less on quadratic equations.

Modern university syllabi: As with school mathematics, the syllabus should be modernized to focus on modelling skills and use of software tools. The current historical sequence should be overturned so that students get to use mathematics on realistic problems before they probe more deeply into the theories and limitations of those methods.

Abstract algebra could be modernized, making it focus on properties that are of more practical relevance, and making it more organized and more of a time saver.

Modules on design and use of practical number systems should be included in the first year. Practical numbers are the systems used for labelling numbers, such as decimal, octal, floating point, and so on. The issues to deal with are precision and management of rounding errors through calculations.

Teach development of mathematics: The true skill of applying mathematics in the real world is much more than just applying the methods taught in school and university. The true skill is the ability to develop applications of mathematics. This should be taught explicitly and intensively.

Development of applications is what happens when a mathematician approaches a complex problem area for the first time, intending not just to solve one little problem but to develop methods and results that will tackle a number of problems in a particular area. It involves deciding what elements to put into models, choosing notation, creating formulae, establishing useful identities and theorems, writing software, then devising processes for making models, testing them, and using them.

This skill involves thinking about what the mathematics will be used for, understanding decision-making processes, considering what data are available, and even devising management processes within which the mathematical methods are used.

Rather than this skill of developing applications being a final stage in the education of mathematicians it should be something that begins in school and then continues to be the main criterion of excellence.

Teach promotion of mathematics: University courses should also teach the skills and importance of promoting mathematics and particular applications of mathematics. Students should graduate with the ability to explain, reassure, coach, and automate mathematics they have applied or developed. They should understand common objections to using mathematical methods and know how to overcome them.

5.4 Applications

Increasing the number and value of applications of mathematics is an intended result of changes to the invention and teaching of mathematics.

However, there are also ways that individuals and organizations can promote useful applications of mathematics, starting now.

Getting over mathophobia: Many people in powerful positions today have forgotten the little mathematics they nearly learned at school. Consequently, maths is one of those things they would rather not deal with, even if they are not, strictly speaking, phobic.

We might imagine that, if we employ someone to do maths in our organization, then working with them will be immensely complicated, hard to understand, not particularly useful, and as much fun as a visit to the dentist.

All that could happen, but with the right management it need not.

Management skills: Using mathematics in an organization when you yourself are not a good mathematician is possible with the right approach. Learn to spot mathematicians who can apply their knowledge to real problems, who can communicate, who can see ways to be useful. Learn to tell the difference between writing about mathematics that is unreasonably unclear and writing that genuinely shares thinking and proves results. Learn to push mathematicians for useful results very soon, and better results soon after that – and learn to spot when they are wasting time. Learn to spot the difference between guesses presented as facts and properly tested models, verified against reality. Learn how software makes mathematics useful in the modern world.

Developing capability: Select and promote people with the ability to develop and apply mathematics in your organization. Evaluate them and suggest the skills they need to improve.

Software: Make good software freely available to more employees and encourage them to at least dabble. Although you can spend a fortune on software it is also possible to spend nothing at all and use the awesome tools developed by and used by academics. An important advantage of doing this is that students you hire will often already know how to use this free software.

6. Summary

Mathematics has already made a huge contribution to the efficiency and beauty of our modern world.

However, it's not hard to see that there are now some huge opportunities to reduce the costs of mathematics and increase its contribution to society. The starting point in many cases is simply to try to do those things, rather than accept defences of the status quo. Why should it be possible for very smart people to get a living without even *trying* to do something useful?

7. References

Leitch, M (2009). *How to write mathematics clearly and keep more readers*. Available at: http://www.learningideas.me.uk/clearmaths/